DETAIL CALCULATION OF "AN L^1-TYPE ESTIMATES FOR RIESZ POTENTIALS"

1. Introduction

In general, the space L^1 is hard to study in many sense. First, if we consider Hölder's inequality, then the natural pair of L^1 is L^∞. In the case of bounded domain, $L^\infty \subset L^p$ for any $p < \infty$ So we can think L^∞ has less elements than any other space. Second, when we study the theory of partial differential equation, Calderón-Zygmund singular integral and Maximal function estimate are useful technique. These technique gives a good way to obtain L^p-estimates. But these technique cannot be applicable to L^1-type estimate. Third, the dual of L^1 is L^∞, but L^1 cannot be considered as a dual of meaningful space. So we cannot apply weak* convergence in this space. By these reasons, many mathematician study a proper subspace H^1, real Hardy space. It has fruitful structure that all disadvantages we discussed do not happen. For further information, see Stein's harmonic analysis or Coifman-Meyer-Lions-Semes's paper.

Currently, there is no general method for studying the space L^1. In 2004, however, Bourgain and Brezis [1] gives a new type estimate which is related to this type of difficulty. They proved the following inequality:

Theorem. *Let $f \in L^1(\mathbb{R}^n;\mathbb{R}^n)$ with $\operatorname{div} f = 0$. For any $u \in W^{1,n}(\mathbb{R}^n;\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we have*

$$\left| \int_{\mathbb{R}^n} f \cdot u dx \right| \le C \|f\|_{L^1} \|\nabla u\|_{L^n}.$$

They proved this inequality by using Littlewood-Paley decomposition. But there is a quite elementary proof given by Van Schaftigen in 2004 [4]. He proved similar inequality in more general setting.

Theorem. *Let $f \in L^1(\mathbb{R}^n;\mathbb{R}^n)$ with $\operatorname{div} f \in L^1$. For any $u \in W^{1,n}(\mathbb{R}^n;\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we have*

$$\left| \int_{\mathbb{R}^n} f \cdot u dx \right| \le C \left(\|f\|_{L^1} \|\nabla u\|_{L^n} + \|\operatorname{div} f\|_{L^1} \|u\|_{L^n} \right).$$

The method is based on dimension reduction and Morrey-Sobolev embedding.

In 2014, Schikorra, Spector and Van Schaftigen [5] proved the following theorem.

Theorem. *Let $n \ge 2$ and $0 < \alpha < n$. Then there exists a constant $C = (\alpha, n) > 0$ such that*

$$\|I_\alpha u\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \le C \|Ru\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$ such that $Ru \in L^1(\mathbb{R}^n;\mathbb{R}^n)$, where R is the vector Riesz transform.

There is another analogue for this result. If we define the Riesz fractional gradient by

$$D^\alpha u := DI_{1-\alpha} u,$$

then we obtain the following result:

Theorem. *Let $n \geq 2$ and $0 < \alpha < 1$. Then there exists a constant $C = (\alpha, n) > 0$ such that*

$$\|u\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \leq C \|D^\alpha u\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

This deals some critical case.

2. Riesz potential, Riesz transforms and Sobolev embedding

2.1. **Riesz potential.**

Let $f \in \mathcal{S}$. Then we have

$$(-\triangle f)^\wedge(\xi) = 4\pi^2 |\xi|^2 \hat{f}(\xi).$$

Movitated by this identity, one may define more general type:

$$\left((-\triangle)^{\frac{\beta}{2}} f\right)^\wedge(\xi) = (2\pi|\xi|)^\beta \hat{f}(\xi).$$

So we define

$$(-\triangle)^{\frac{\beta}{2}} f(x) := \int_{\mathbb{R}^n} (2\pi|\xi|)^\beta \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi.$$

Here the operator $(-\triangle)^{\frac{\beta}{2}}$ acts as a derivative of order β if β is a positive integer. Note that $|\xi|^\beta \notin L^1_{\text{loc}}$ if $\beta < -n$. So we assume $\beta > -n$ or $\beta \leq -n$ and $\hat{f}$ vanishes to sufficiently high order at the origin so that $|\xi|^\beta \hat{f} \in L^1_{\text{loc}}$.

So we are ready to define the Riesz potential, which would be an inverse of fractional Laplacian.

Definition 2.1. Let $0 < \alpha < n$. The Riesz potential of order α is the operator

$$I_\alpha = (-\triangle)^{-\frac{\alpha}{2}}$$

and

$$I_\alpha(f)(x) = 2^{-\alpha}\pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} f(x-y)|y|^{-n+\alpha} dy.$$

Note that the integral is convergent if $f \in \mathcal{S}$.

By calculation, it is easy to verify

Proposition 2.2. *The identity $(I_\alpha f)^\wedge(\xi) = (2\pi|\xi|)^{-\alpha}\hat{f}(\xi)$ holds in the sense that*

$$\int_{\mathbb{R}^n} I_\alpha(f)(x)\overline{g(x)}dx = \int_{\mathbb{R}^n} \hat{f}(\xi)(2\pi|\xi|)^{-\alpha}\overline{\hat{g}(\xi)}d\xi$$

whenever $f, g \in \mathcal{S}$.

Proof. See Stein's singular integral [7]. □

Riesz potential satisfies semigroup property.

Proposition 2.3. *The following identity holds:*

$$I_\alpha(I_\beta f) = I_{\alpha+\beta} f \quad \text{for } f \in \mathcal{S}(\mathbb{R}^n)$$

when $\alpha, \beta > 0$ and $\alpha + \beta < n$.

Proposition 2.4. *For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$I_\alpha\left((-\triangle)^{\frac{\alpha}{2}}\varphi\right) = \varphi.$$

Proposition 2.5. *For any* $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ *and* $0 < \alpha < n$, *we have*

$$\int_{\mathbb{R}^n} \varphi\psi dx = \int_{\mathbb{R}^n} (I_\alpha \psi)(-\triangle)^{\frac{\alpha}{2}} \varphi dx.$$

Proof. By proposition 2.4, we have

$$\int_{\mathbb{R}^n} \varphi\psi dx = \int_{\mathbb{R}^n} I_\alpha \left((-\triangle)^{\frac{\alpha}{2}} \varphi\right) \psi dx.$$

Then by Fubini's theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^n} I_\alpha \left((-\triangle)^{\frac{\alpha}{2}} \varphi\right) \psi dx &= \int_{\mathbb{R}^n} \left[\frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{(-\triangle)^{\frac{\alpha}{2}} \varphi(y)}{|x-y|^{n-\alpha}} dy\right] \psi(x)\, dx \\ &= \int_{\mathbb{R}^n} \left[\frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{\psi(x)}{|y-x|^{n-\alpha}} dx\right] (-\triangle)^{\frac{\alpha}{2}} \varphi(y)\, dy \\ &= \int_{\mathbb{R}^n} I_\alpha(\psi)(-\triangle)^{\frac{\alpha}{2}} \varphi dx.\end{aligned}$$

So we are done. □

The following theorem is important when we derive Sobolev embedding theorem. This is so called Hardy-Littlewood-Sobolev inequality:

Theorem 2.6. *For* $f \in L^p(\mathbb{R}^n)$, *we have*

$$\|I_\alpha(f)\|_q \leq C_{p,q} \|f\|_p$$

where $0 < \alpha < n$ *and* $1 < p < \infty$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{n-\alpha}{n}$.

Proof. For $f \in \mathcal{S}(\mathbb{R}^n)$, fix $0 < R < \infty$. Write

$$\begin{aligned}\left(f * |y|^{-\alpha}\right)(x) &= \int f(x-y)\,|y|^{-\alpha}\, dy \\ &= \int_{|y|<R} f(x-y)\,|y|^{-\alpha}\, dy + \int_{|y|\geq R} f(x-y)\,|y|^{-\alpha}\, dy.\end{aligned}$$

First,

$$\int_{|y|<R} f(x-y)\,|y|^{-\alpha}\, dy = \int f(x-y)\,|y|^{-\alpha} \chi_R(y)\, dy.$$

Since $|y|^{-\alpha} \chi_R(y)$ is radial and decreasing and integrable since $\alpha < n$, so we can majorize the integral by

$$\lesssim (Mf)(x) \int_{|y|\leq R} |y|^{-\alpha}\, dy = CR^{n-\alpha}(Mf)(x).$$

Also Hölder's inequality gives

$$\int_{|y|\geq R} f(x-y)\,|y|^{-\alpha}\, dy \leq \|f\|_p \left\| |y|^{-\alpha} \chi_{B_R^c} \right\|_{p'}.$$

Note that $\alpha p' > n$ since $\alpha p' - n = \frac{np'}{q} > 0$. So

$$\left\| |y|^{-\alpha} \chi_{B_R^c} \right\|_{p'} = CR^{-\frac{n}{q}}.$$

Hence

$$|I_\alpha(f)(x)| \lesssim \left[Mf(x)\, R^{n-\alpha} + \|f\|_p\, R^{-\frac{n}{q}}\right].$$

Since $R > 0$ was arbitrary, we can choose R so that

$$|I_\alpha(f)(x)| \lesssim [Mf(x)]^{\frac{p}{q}} \|f\|_p^{1-\frac{p}{q}}.$$

Hence by L^p-boundedness of maximal function, we obtain

$$\|I_\alpha(f)\|_q \le C\|f\|_p.$$

This is true for any $f \in \mathcal{S}$. Hence by density argument, we are done. □

Remark. When $p = 1$, the inequalty

$$\|I_\alpha(f)\|_{\frac{n}{n-\alpha}} \le C\|f\|_1$$

cannot hold.

If it were true, then we put in the place of f a sequence $\{f_n\}$ of positive integrable functions whose common integral is one and whose supports converges to the origin. Then with f_m, we have

$$I_\alpha(f_m)(x) \to \frac{1}{\gamma(\alpha)} \frac{1}{|x|^{n-\alpha}}.$$

If $\|I_\alpha f_m\|_{\frac{n}{n-\alpha}} \le C$ were valid uniformly in m, then Fatou's lemma gives

$$\int_{\mathbb{R}^n} \frac{1}{|x|^{(n-\alpha)\frac{n}{n-\alpha}}} dx < \infty,$$

a contradiction.

2.2. **Sobolev embeddings.** The importance of the Sobolev spaces just considered is that in terms of them we can account in a relatively simple way how restrictions on the "size" of partial derivatives imply corresponding restrctions on the functions in questions.

Theorem 2.7 (Sobolev's embedding theorem)**.** *Suppose k is a positive integer, and $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$.*

(a) *If $q < \infty$ (i.e. $p < \frac{n}{k}$), then $W^{k,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ and the natural inclusion map is continuous.*
(b) *If $q = \infty$ (i.e. $p = \frac{n}{k}$), then the restriction of an $f \in W^{k,p}(\mathbb{R}^n)$ to a compact subset of $\mathbb{R}^n$ belongs to $L^r(\mathbb{R}^n)$ for every $r < \infty$.*
(c) *If $p > \frac{n}{k}$, then every $f \in W^{k,p}(\mathbb{R}^n)$ can be modified on a set of zero measure so that the resulting function is continuous.*

Let us proceed in a purely formal way, operating always with functions of the class $\mathcal{S}$. Let $f \in \mathcal{S}$. Then the Fourier transform of $\frac{\partial f}{\partial x_j}$ is $(2\pi i\xi_j)\hat{f}(\xi)$. Recall the jth Riesz transform R_j. Then we have

$$\begin{aligned}\left(R_j\left(\frac{\partial}{\partial x_j} f\right)\right)^\wedge(\xi) &= -\frac{i\xi_j}{|\xi|}\left(\frac{\partial}{\partial x_j} f\right)^\wedge(\xi) \\ &= \frac{2\pi\xi_j^2}{|\xi|}\hat{f}(\xi).\end{aligned}$$

This means

$$\left(\sum_{j=1}^n R_j\left(\frac{\partial}{\partial x_j} f\right)\right)^\wedge(\xi) = (2\pi|\xi|)\hat{f}(\xi),$$

or

$$\hat{f}(\xi) = (2\pi|\xi|)^{-1}\left(\sum_{j=1}^{n} R_j\left(\frac{\partial}{\partial x_j} f\right)\right)^{\wedge}(\xi)$$

So

$$I_1\left(\sum_{j=1}^{d} R_j\left(\frac{\partial}{\partial x_j} f\right)\right) = f. \tag{2.1}$$

The above identity shows f is represented by the Riesz transform and the potential of order 1. Note that the Riesz transforms is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and the potential of order 1 maps $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for appropriates p and q. But we use another approach which is closely related to the identity (2.1) but avoids the use of the rather deep theory of the Riesz transform.

Proof. We first assume that $k = 1$.

(a) We will show

$$f(x) = \frac{1}{\omega_{n-1}} \sum_{j=1}^{n} \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x-y)\,\frac{y_j}{|y|^n}\,dy, \tag{2.2}$$

where ω_{n-1} is the area of the sphere $\mathbb{S}^{n-1}$.

If $f \in \mathscr{D}(\mathbb{R}^1)$, then we have

$$f(x) = [-f(x-t)]_0^{\infty} = \int_0^{\infty} f'(x-t)\,dt.$$

Its n-dimensional analogue is

$$f(x) = \int_0^{\infty} (\nabla f(x-\xi t), \xi)\,dt$$

where $\xi \in \mathbb{S}^{n-1}$ and ∇f is the vector with components

$$\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right).$$

To verify it more easily, fix any $\xi \in \mathbb{S}^{n-1}$. Define $h(t) = x - t\cdot\xi$. Then $f \circ h$ is one-variable function. So

$$(f\circ h)'(t) = \nabla f(x - t\cdot\xi)\cdot(-\xi).$$

This implies

$$\begin{aligned}&\int_0^{\infty} \nabla f(x-t\xi)\cdot(-\xi)\,dt\\ &= \lim_{t\to\infty}(f\circ h)(t) - (f\circ h)(0)\\ &= -f(x).\end{aligned}$$

Hence

$$f(x) = \int_0^{\infty} \nabla f(x-t\xi)\cdot\xi dt.$$

Then by integrating $\xi \in \mathbb{S}^{n-1}$ ranging on the unit sphere, we have

$$f(x) = \frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}}\int_0^{\infty} (\nabla f(x-\xi t), \xi)\,dtd\sigma(\xi)$$

$$= \frac{1}{\omega_{n-1}} \int_0^\infty \int_{\mathbb{S}^{n-1}} \sum_{j=1}^n \frac{\partial f}{\partial x_j}(x - \xi t)\, \xi_j d\sigma(\xi)\, dt$$
$$= \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \sum_{j=1}^n \frac{\partial f}{\partial x_j}(x - y)\, \frac{y_j}{|y|^n} dy.$$

Note that $f = -I_2(\triangle f)$ by Proposition 2.3.

Another useful observation is the following: if $F = I_1(f)$, then $\frac{\partial F}{\partial x_j} = -R_j(f)$. Indeed,

$$\int_{\mathbb{R}^n} I_1(f)(x)\, \overline{\frac{\partial \varphi}{\partial x_j}} dx = \int_{\mathbb{R}^n} (2\pi |x|)^{-1} \hat{f}(x)\, \overline{\left(\frac{\partial \varphi}{\partial x_j}\right)^{\wedge}(x)} dx$$
$$= \int_{\mathbb{R}^n} (2\pi |x|)^{-1} \hat{f}(x)\, \overline{(2\pi i x_j)\, \hat{\varphi}(x)} dx$$
$$= \int_{\mathbb{R}^n} \hat{f}(x) \left(-i \frac{x_j}{|x|}\right) \overline{\hat{\varphi}(x)} dx$$
$$= \int_{\mathbb{R}^n} R_j(f)(x)\, \overline{\varphi} dx$$

and therefore $\frac{\partial I_1(f)}{\partial x_j} = R_j(f)(x)$ in a distributional sense when $f \in \mathcal{S}(\mathbb{R}^n)$.

So we prove the theorem first in the case $k = 1$, and where $1 < p, q < \infty$. Assume $f \in C_0^\infty$. Then (2.2) shows that

$$|f(x)| \le A \sum_{j=1}^n \int_{\mathbb{R}^n} \left| \frac{\partial f}{\partial x_j}(x - y) \right| \frac{1}{|y|^{n-1}} dy. \tag{2.3}$$

Therefore by Theorem 2.6, (the case $\alpha = 1$), we get

$$\|f\|_q \le A' \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_p, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{d}. \tag{2.4}$$

Let now $f \in W^{1,p}(\mathbb{R}^n)$. Then there exists a sequence of elements of C_0^∞, $\{f_n\}$ so that $f_k \to f$ in L^p norm, and $\frac{\partial f_k}{\partial x_j}$ converges in L^p norm. The limit $\lim_{k\to\infty} \frac{\partial f_k}{\partial x_j} = \frac{\partial f}{\partial x_j}$ since

$$\int_{\mathbb{R}^n} f \frac{\partial \varphi}{\partial x_j} dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k \frac{\partial \varphi}{\partial x_j} dx$$
$$= -\lim_{k\to\infty} \int_{\mathbb{R}^n} \frac{\partial f_k}{\partial x_j} \varphi dx$$
$$= -\int_{\mathbb{R}^n} \lim_{k\to\infty} \frac{\partial f_k}{\partial x_j} \varphi dx$$

for every $\varphi \in C_0^\infty$. So by (2.4), we get

$$\|f_k - f_m\|_q \le A' \sum_{j=1}^n \left\| \frac{\partial f_k}{\partial x_j} - \frac{\partial f_m}{\partial x_j} \right\|_p,$$

and so the sequence $\{f_k\}$ also converges in $L^q(\mathbb{R}^n)$ norm, and this limit must also equal f. Thus $f \in L^q(\mathbb{R}^n)$, and

$$\|f\|_q \le A' \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_p \le A' \|f\|_{1,p}, \quad f \in W^{1,p}(\mathbb{R}^n).$$

This shows that $f \in L^q(\mathbb{R}^n)$ and the inclusion mapping of $W^{1,p}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ is continuous.

(b) We consider next the situation when $k = 1$ but $q = \infty$or $p > \frac{n}{k}$. The relevant conclusions of Theorem 2.7 are local in character and so we may simplify matters by reducing to the case when f have compact support. Given any fixed compact set K, let η be a function in C_0^∞ which is one on that set. If $f \in W^{1,p}(\mathbb{R}^n)$, consider $\eta \cdot f$.

It will be enough to prove the conclusions (ii) and (iii) for ηf, which incidentally also belongs to $W^{1,p}(\mathbb{R}^n)$.

To see that $\eta f \in W^{1,p}(\mathbb{R}^n)$, it suffices to verify that the derivative $\frac{\partial}{\partial x_j}(\eta f) = \frac{\partial \eta}{\partial x_j} f + \eta \frac{\partial f}{\partial x_j}$ in the weak sense. However,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\partial}{\partial x_j}(\eta f)\,\varphi dx \\
&= -\int_{\mathbb{R}^n} (\eta f)\frac{\partial \varphi}{\partial x_j} dx \\
&= -\int_{\mathbb{R}^n} f\frac{\partial(\varphi\eta)}{\partial x_j} dx + \int_{\mathbb{R}^d} \varphi f \frac{\partial \eta}{\partial x_j} dx \\
&= \int_{\mathbb{R}^n} \left(\frac{\partial \eta}{\partial x_j} f + \eta \frac{\partial f}{\partial x_j}\right)\varphi dx
\end{aligned}$$

and this assertion is proved.

We start therefore with $f \in W^{1,p}(\mathbb{R}^n)$ and its approximating sequence $\{f_n\}$. Choose an R so large that if K_1 is the compact support of η, then the set $K \setminus K_1$ is contained in the ball of radius R about the origin.

Then by (2.3), we get

$$|\eta(x) f_k(x)| \le A \sum_{j=1}^{n} \int_{|y| \le R} \left| \frac{\partial(\eta f_k)}{\partial x_j}(x-y) \right| |y|^{-n+1} dy, \quad x \in K. \tag{2.5}$$

Recall the Young’s inequality for convolution which states that if

$$\mathscr{C}(x) = \int_{\mathbb{R}^d} \mathscr{A}(x-y)\,\mathscr{B}(y)\,dy,$$

then $\|\mathscr{C}\|_r \le \|\mathscr{A}\|_p \|\mathscr{B}\|_s$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{s} - 1$. We set

$$\begin{aligned}
\mathscr{A}(x) &= A \sum_{j=1}^{n} \left| \frac{\partial}{\partial x_j}(\eta f_k) \right| \\
\mathscr{B}(y) &= \begin{cases} \frac{1}{|y|^{n-1}} & \text{if } |y| \le R \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Let $s < \frac{n}{n-1}$. Then $\|\mathscr{B}\|_s < \infty$. So by Young’s inequality, we have

$$\begin{aligned}
&\int_K |f_k|^r dx \\
&\le \int |\eta f_k|^r dx \\
&\le \|\mathscr{C}\|_r^r
\end{aligned}$$

$$\leq A' \left\| \sum_{j=1}^{n} \left| \frac{\partial (\eta f_k)}{\partial x_j} \right| \right\|_p^r .$$

Similarly,

$$\int_K |f_k - f_m|^r \, dx \leq A' \left\| \sum_{j=1}^{n} \left| \frac{\partial (\eta (f_k - f_m))}{\partial x_j} \right| \right\|_p^r .$$

So we see that $\{f_k\}$, which converges to f in the L^p norm, also converges in the L^r norm, when restricted to the set K. Thus $f \in L^r$ when restricted to the set K.

(iii) Note that

$$\sup_x |\mathscr{C}(x)| \leq \|\mathscr{A}\|_p \|\mathscr{B}\|_{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

which follows from Hölder's inequality. Notice that if $p > n$, then

$$\|\mathscr{B}\|_{p'}^{p'} = \int_{|y| \leq R} \frac{1}{|y|^{(n-1)p'}} dy < \infty.$$

So we get

$$\sup_{x \in K} |f_k(x) - f_n(x)| \leq A' \left\| \sum_{j=1}^{n} \left| \frac{\partial \eta (f_k - f_n)}{\partial x_j} \right| \right\|_p .$$

So the continuous functions $\{f_m(x)\}$ converges uniformly on every compact set and hence f may be taken to be continuous.

So far we proved our theorem with assumptions that $k = 1$ and $p > 1$. The method needs to be different since our method was come from the Theorem 2.6. We first prove the following inequality and gives a proof of the case $p = 1$:

$$\|f\|_q \leq \left(\prod_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_1 \right)^{\frac{1}{n}}, \quad \frac{1}{q} = 1 - \frac{1}{n}, \quad f \in C_0^\infty. \tag{2.6}$$

We prove (2.6) by induction on n. The case $n = 1$ is trivial because $f(x) = \int_{-\infty}^{x} f'(t) \, dt$.

We assume therefore that the inequality (2.6) holds for $n - 1$. To use the induction hypothesis, we write $x \in \mathbb{R}^n$ as $x = (x_1, x')$ where $x' \in \mathbb{R}^{n-1}$ and $x_1 \in \mathbb{R}$. We set

$$I_j(x_1) = \int_{\mathbb{R}^{n-1}} \left| \frac{\partial f}{\partial x_j}(x_1, x') \right| dx', \quad j = 2, \ldots, n,$$

and

$$I_1(x') = \int_{\mathbb{R}^1} \left| \frac{\partial f}{\partial x_1}(x_1, x') \right| dx_1.$$

Suppose $q = \frac{n}{n-1}$ and $q' = \frac{n-1}{n-2}$. Then by the case $n - 1$, we have

$$\left(\int_{\mathbb{R}^{n-1}} |f(x_1, x')|^q \, dx' \right)^{\frac{1}{q'}} \leq \left(\prod_{j=2}^{n} I_j(x_1) \right)^{\frac{1}{n-1}} . \tag{2.7}$$

Note that $|f(x)| \leq I_1(x')$ by one-dimensional case since

$$|f(x)| \leq \int_{-\infty}^{x_1} \left| \frac{\partial f}{\partial t_1}(t_1, x_2, \ldots, x_n) \right| dt_1 \leq I_1(x') .$$

So $|f|^q \le (I_1(x'))^{\frac{1}{n-1}} |f|$ since
$$q = \frac{1}{n-1} + 1.$$

Thus by Hölder's inequality, we have
$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} |f|^q \, dx' \\
&\le \int_{\mathbb{R}^{n-1}} (I_1(x'))^{\frac{1}{n-1}} |f| \, dx' \\
&\le \left(\int_{\mathbb{R}^{n-1}} I_1(x') \, dx' \right)^{\frac{1}{n-1}} \left(\int_{\mathbb{R}^{n}} |f|^{q'} \, dx' \right)^{\frac{1}{q'}}.
\end{aligned}$$

Hence by (2.7), we have
$$\int_{\mathbb{R}^{n-1}} |f|^q \, dx' \le \left(\int_{\mathbb{R}^{n-1}} I_1(x') \, dx' \right)^{\frac{1}{n-1}} \left(\prod_{j=2}^{n} I_j(x_1) \right)^{\frac{1}{n-1}}.$$

Integrate this with respect to x_1. Then by multiple Hölder's inequality, we have
$$\begin{aligned}
\int_{\mathbb{R}^{n}} |f|^q \, dx &\le \left(\int_{\mathbb{R}^{n-1}} I_1(x') \, dx' \right)^{\frac{1}{n-1}} \int_{\mathbb{R}^{1}} \left(\prod_{j=2}^{n} I_j(x_1) \right)^{\frac{1}{n-1}} dx_1 \\
&\le \left(\prod_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_1 \right)^{\frac{1}{n-1}}
\end{aligned}$$
which is the desired inequality (2.6) since $q = \frac{n}{n-1}$.

If we recall the arithmetic-harmonic mean inequality,
$$\left(\prod_{j=1}^{n} a_j \right)^{\frac{1}{n}} \le \frac{1}{n} \sum_{j=1}^{n} a_j, \quad \text{if } a_j \ge 0,$$
then as a consequence of (2.6) we have
$$\|f\|_q \le \frac{1}{n} \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_1, \quad f \in C_0^\infty, \quad \frac{1}{q} = 1 - \frac{1}{n}.$$
This result shows $W^{1,1}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ and the inclusion map is continuous.

So far we proved the case $k = 1$. We can argue by induction and show that the case of $k \ge 2$ can be reduced to the case $k = 1$. Let us take the assertion (i) of the theorem. The assumption $f \in W^{k,p}(\mathbb{R}^n)$ clearly implies that $f \in W^{k-1,p}(\mathbb{R}^n)$ and $\frac{\partial f}{\partial x_j} \in W^{k-1,p}(\mathbb{R}^n)$. Hence the case of the theorem for $k-1$ implies that $f \in L^{q'}(\mathbb{R}^n)$ and $\frac{\partial f}{\partial x_j} \in L^{q'}(\mathbb{R}^n)$, where $\frac{1}{q'} = \frac{1}{p} - \frac{(k-1)}{n}$. The case $k = 1$ implies that $f \in L^q(\mathbb{R}^n)$ with
$$\frac{1}{q} = \frac{1}{q'} - \frac{1}{n} = \frac{1}{p} - \left(\frac{k-1}{n} \right) - \frac{1}{n} = \frac{1}{p} - \frac{k}{n}.$$
The corresponding inclusion mappings are also continuous. This completes the proof. □

2.3. **Riesz transforms.** Introduce tempered distributions W_j on $\mathbb{R}^n$, for $1 \leq j \leq n$, as follows: for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, let

$$\langle W_j, \varphi \rangle = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{y_j}{|y|^{n+1}} \varphi(y)\, dy.$$

Note that $W_j \in \mathcal{S}'(\mathbb{R}^n)$. From this, we define the jth Riesz transform.

Definition 2.8. For $1 \leq j \leq n$, the jth *Riesz transform* of f is given by convolution with the distribution W_j, that is,

$$R_j(f)(x) = (f * W_j)(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \text{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy,$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

Proposition 2.9. *The jth Riesz transform R_j is given on the Fourier transform side by multiplication by the function $-i\frac{\xi_j}{|\xi|}$. That is, for any f in $\mathcal{S}(\mathbb{R}^n)$, we have*

$$R_j(f)(x) = \left(-\frac{i\xi_j}{|\xi|}\hat{f}(\xi)\right)^{\vee}(x).$$

Proposition 2.10. *The Riesz transforms satisfy*

$$-I = \sum_{j=1}^{n} R_j^2,$$

where I is the identity operator.

Theorem 2.11. *Let f and $f_1, \ldots, f_n$ all belong to $L^2(\mathbb{R}^n)$, and let their respective Poisson integrals be $u_0(x,y) = (\mathcal{P}_y * f)(x)$, $u_1(x,y) = (\mathcal{P}_y * f_1)(x), \ldots, u_n(x,y) = (\mathcal{P}_y * f_n)(x)$. Then a necessary and sufficient condition that*

$$f_j = R_j(f), \quad j = 1, \ldots, n,$$

is that the following generalized Cauchy-Riemann equations hold:

$$\begin{cases} \sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0, \\ \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \quad j \neq k. \end{cases}$$

There is another characterization on jth Riesz transform.

Theorem 2.12. *For any $f \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$R_j f = D_j I_1 f.$$

3. Proof of the theorem

In this section, we prove the main theorem. All details will be presented. Before to start the proof, we introduce some notations. Since we consider whole domain $\mathbb{R}^n$, we denote $\|\cdot\|_p$ as $\|\cdot\|_{L^p(\mathbb{R}^n)}$ or $\|\cdot\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}$.

Let $u \in C_0^\infty(\mathbb{R}^n)$ be such that $Ru \in L^1(\mathbb{R}^n;\mathbb{R}^n)$. First, if we assume the following inequality

$$\left| \int_{\mathbb{R}^n} R_j u \varphi dx \right| \leq C \|Ru\|_1 \left\| (-\triangle)^{\frac{\alpha}{2}} \varphi \right\|_{\frac{n}{\alpha}} \tag{3.1}$$

for every $\varphi \in C_0^\infty$, the theorem follows.

Indeed, by proposition 2.10 $v = -\sum_{j=1}^{n} R_j^2 v$ and the L^p-boundedness of R_i, we have

$$\begin{aligned}\|I_\alpha u\|_{\frac{n}{n-\alpha}} &\leq C \sum_{i=1}^{n} \|R_i I_\alpha u\|_{\frac{n}{n-\alpha}} \\ &= C \sum_{i=1}^{n} \sup_{\|\psi_i\|_{\frac{n}{\alpha}} \leq 1} \int_{\mathbb{R}^n} R_i I_\alpha u \psi_i dx.\end{aligned}$$

Note that for any $f \in L^1_{\text{loc}}$ and $g \in C_0^k$, we have $f * g \in C^k$ and

$$D^\alpha (f * g)(x) = (f * D^\alpha g)(x) \quad \text{if } |\alpha| \leq k$$

So by this,

$$\begin{aligned}\int_{\mathbb{R}^n} R_i (I_\alpha u) \psi_i dx &= \int_{\mathbb{R}^n} D_i I_1 (I_\alpha u) \psi_i dx \\ &= \int_{\mathbb{R}^n} I_1 (I_\alpha D_i u) \psi_i dx.\end{aligned}$$

If $\alpha + 1 < n$, then we have

$$\begin{aligned}\int_{\mathbb{R}^n} I_1 (I_\alpha D_i u) \psi_i dx &= \int_{\mathbb{R}^n} I_\alpha (I_1 D_i u) \psi_i dx \\ &= \int_{\mathbb{R}^n} I_\alpha (D_i I_1 u) \psi_i dx.\end{aligned}$$

Then by Fubini's theorem, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \frac{1}{\gamma(\alpha)} \frac{D_i I_1 u(y)}{|x-y|^{n-\alpha}} dy\right] \psi_i(x)\, dx \\ &= \int_{\mathbb{R}^n} D_i I_1 u(y) \left[\int_{\mathbb{R}^n} \frac{1}{\gamma(\alpha)} \frac{\psi_i(x)}{|x-y|^{n-\alpha}} dx\right] dy \\ &= \int_{\mathbb{R}^n} R_i u(y)\, I_\alpha(\psi_i)(y)\, dy.\end{aligned}$$

If $1 \leq n-1 \leq \alpha < n$, by semigroup property of Riesz potential and Fubini's theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^n} I_1 (I_\alpha D_i u) \psi_i dx &= \int_{\mathbb{R}^n} I_1 (I_{\alpha-1} I_1 D_i u) \psi_i dx \\ &= \int_{\mathbb{R}^n} (I_1 I_{\alpha-1} D_i u)(I_1 \psi_i)\, dx \\ &= \int_{\mathbb{R}^n} (I_{\alpha-1} I_1 D_i u)(I_1 \psi_i)\, dx \\ &= \int_{\mathbb{R}^n} (I_1 D_i u)(I_{\alpha-1} I_1 \psi_i)\, dx \\ &= \int_{\mathbb{R}^n} (R_i u)\, I_\alpha \psi_i dx.\end{aligned}$$

Hence we obtain

$$\int_{\mathbb{R}^n} R_i (I_\alpha u) \psi_i dx = \int_{\mathbb{R}^n} (R_i u)(I_\alpha \psi_i)\, dx.$$

Therefore, density argument allows us to take $\varphi = I_\alpha \psi_i$. So by (3.1), we have

$$\begin{aligned}\|I_\alpha u\|_{\frac{n}{n-\alpha}} &\le C\sum_{i=1}^{n}\sup_{\|\psi_i\|_{\frac{n}{\alpha}}\le 1}\|Ru\|_1\left\|(-\triangle)^{\frac{\alpha}{2}} I_\alpha\psi_i\right\|_{\frac{n}{\alpha}}\\ &= C\sum_{i=1}^{n}\sup_{\|\psi_i\|_{\frac{n}{\alpha}}\le 1}\|Ru\|_1\|\psi_i\|_{\frac{n}{\alpha}}\\ &\le C\|Ru\|_1\end{aligned}$$

Now we prove the inequality. Introduce a family of molifiers: we take $\rho \in C_0^\infty(\mathbb{R}^n)$ such that supp$\rho \subset B(0,1)$ and $\int_{\mathbb{R}^n}\rho = 1$. We define $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$ and $\varphi_\varepsilon(x) = (\varphi * \rho_\varepsilon)(x)$.

Fix $j = 1$ and write $x = (x', x_n)$. Then

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}} R_1 u(x', x_n)\varphi(x', x_n)\,dx'\\ &= \int_{\mathbb{R}^{n-1}} R_1 u(x', x_n)\left[\varphi(x', x_n) - \varphi_\varepsilon(x', x_n)\right]dx'\\ &\quad + \int_{\mathbb{R}^{n-1}} R_1 u(x', x_n)\varphi_\varepsilon(x', x_n)\,dx'\\ &=: I(\varepsilon) + II(\varepsilon).\end{aligned}$$

For $I(\varepsilon)$, Hölder's inequality gives

$$I(\varepsilon) \le \|Ru_1(\cdot, x_n)\|_{L^1(\mathbb{R}^{n-1})}\|\varphi(\cdot, x_n) - \varphi_\varepsilon(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})}.$$

Now the fundamental theorem of calculus gives

$$\begin{aligned}\varphi_\varepsilon(x', x_n) - \varphi_\delta(x', x_n) &= \int_\delta^\varepsilon \frac{\partial}{\partial r}\varphi_r(x', x_n)\,dr\\ &= \int_\delta^\varepsilon\int_{\mathbb{R}^n}\sigma_r(x-y)\varphi(y)\,dydr.\end{aligned}$$

Here

$$\sigma_r(z) := \frac{\partial\rho_r}{\partial r}(z) = \frac{1}{r^n}\left[-\nabla\rho\left(\frac{z}{r}\right)\cdot\frac{z}{r^2} - \frac{n}{r}\rho\left(\frac{z}{r}\right)\right].$$

For fixed x, note that

$$\varphi_\varepsilon(x) - \varphi_\delta(x) = \int_{|z|<1}\rho(z)\left[\varphi(x-\varepsilon z) - \varphi(x-\delta z)\right]dz.$$

So

$$|\varphi_\varepsilon(x) - \varphi_\delta(x)| \le \|\nabla\varphi\|_{L^\infty}(\varepsilon - \delta).$$

This shows $\left|\frac{\partial}{\partial r}\varphi_r(x)\right| \le C$. So by letting $\delta \to 0$, dominated convergence theorem gurantees

$$\varphi_\varepsilon(x', x_n) - \varphi(x', x_n) = \int_0^\varepsilon\int_{\mathbb{R}^n}\sigma_r(x-y)\varphi(y)\,dydr.$$

Then by proposition 2.5, we have

$$\varphi_\varepsilon(x', x_n) - \varphi(x', x_n) = \int_0^\varepsilon\int_{\mathbb{R}^n} I_\alpha\sigma_r(x-y)(-\triangle)^{\frac{\alpha}{2}}\varphi(y)\,dydr.$$

We claim that

$$|I_\alpha\sigma_r(z)| \le \frac{C}{(r+|z|)^{n-\alpha+1}}. \tag{3.2}$$

When $|z| \leq 2r$

$$\begin{aligned}|I_\alpha \sigma_r(z)| &= \left| \frac{C}{r^n} \int_{B(0,r)} \frac{\nabla \rho\left(\frac{z}{r}\right) \cdot \frac{z}{r^2} + \frac{n}{r}\rho\left(\frac{z}{r}\right)}{|z-y|^{n-\alpha}} dy \right| \\ &\leq \frac{C}{r^{n+1}} \int_{B(0,r)} \frac{1}{|z-y|^{n-\alpha}} dy.\end{aligned}$$

Since $|z| \leq 2r$, $|z-y| \leq |z| + |y| \leq 3r$. So the above integral is less than or equal to

$$\leq \frac{C}{r^{n+1}} \int_{B(0,3r)} \frac{1}{|y|^{n-\alpha}} dy = \frac{C}{r^{n-\alpha+1}}.$$

Since $|z| \leq 2r$, $(|z|+r)^{n-\alpha+1} \leq 3^{n-\alpha+1} r^{n-\alpha+1}$. So

$$|I_\alpha \sigma_r(z)| \lesssim \frac{1}{(|z|+r)^{n-\alpha+1}}.$$

For any scalar function p and a vector field $\mathbb{F}$, we have

$$\text{div } (p\mathbb{F}) = p \text{div } \mathbb{F} + \nabla p \cdot \mathbb{F}.$$

So

$$\begin{aligned}\text{div } \left(\rho\left(\frac{y}{r}\right)\frac{y}{r}\right) &= \rho\left(\frac{y}{r}\right) \text{div } \left(\frac{y}{r}\right) + \frac{1}{r}\nabla\rho\left(\frac{y}{r}\right) \cdot \frac{y}{r} \\ &= \frac{n}{r}\rho\left(\frac{y}{r}\right) + \frac{1}{r^2}\nabla\rho\left(\frac{y}{r}\right) \\ &= -\sigma_r(y).\end{aligned}$$

So

$$I_\alpha \sigma_r(z) = -\frac{C}{r^n} \int_{B(0,r)} \frac{\text{div } \left(\rho\left(\frac{y}{r}\right)\frac{y}{r}\right)}{|z-y|^{n-\alpha}} dy.$$

Since $\text{supp}(\rho) \subset B(0,r)$ and $\nabla\left(|z|^{-\beta}\right) = -\beta \frac{1}{|z|^{\beta+1}} \frac{z}{|z|}$, integration by part gives

$$I_\alpha \sigma_r(z) = \frac{C}{r^n} \int_{B(0,r)} \frac{\rho\left(\frac{y}{r}\right)\frac{y}{r}}{|z-y|^{n-\alpha+1}} \cdot \frac{y-z}{|y-z|} dy.$$

So the change of variable $w = \frac{y}{r}$ yields the bound

$$\begin{aligned}|I_\alpha \sigma_r(z)| &\leq \int_{B(0,1)} \frac{C}{|z-rw|^{n-\alpha+1}} dw \\ &= \frac{1}{|z|^{n-\alpha+1}} \int_{B(0,1)} \frac{C}{\left|\frac{z}{|z|} - \frac{r}{|z|}w\right|^{n-\alpha+1}} dw.\end{aligned}$$

For $|z| \geq 2r$,

$$\int_{B(0,1)} \frac{C}{\left|\frac{z}{|z|} - \frac{r}{|z|}w\right|^{n-\alpha+1}} dw \leq C$$

since $\frac{C}{\left|\frac{z}{|z|} - \frac{r}{|z|}w\right|^{n-\alpha+1}}$ is bounded on $B(0,1)$. Since $|z| \geq 2r$, $\frac{1}{|z|^{n-\alpha+1}} \leq \frac{C}{(|z|+r)^{n-\alpha+1}}$, so we are done.

Therefore, we can estimate

$$\|\varphi(\cdot, x_n) - \varphi_\varepsilon(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \leq C \sup_{x' \in \mathbb{R}^{n-1}} \int_0^\varepsilon \int_{\mathbb{R}^n} \frac{\left|(-\triangle)^{\frac{\alpha}{2}} \varphi(y)\right|}{\left(r + |x-y|^{n-\alpha+1}\right)} dy dr.$$

By the Hölder's inequality on $\mathbb{R}^{n-1}$, we deduce that

$$\begin{aligned}&\|\varphi(\cdot,x_n)-\varphi_\varepsilon(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\\&\le C\sup_{x'\in\mathbb{R}^{n-1}}\int_0^\varepsilon\int_{\mathbb{R}}\left(\int_{\mathbb{R}^{n-1}}\left|(-\triangle)^{\frac{\alpha}{2}}\varphi(y',y_n)\right|^{\frac{n}{\alpha}}dy'\right)^{\frac{\alpha}{n}}\\&\times\left(\int_{\mathbb{R}^{n-1}}\frac{1}{\left(r+\sqrt{|x_n-y_n|^2+|x'-y'|^2}\right)^{n+\frac{n}{n-\alpha}}}dy'\right)^{1-\frac{\alpha}{n}}\quad dy_n dr.\end{aligned}$$

Scaling argument gives

$$\sup_{x'\in\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}^{n-1}}\frac{1}{\left(r+\sqrt{|x_n-y_n|^2+|x'-y'|^2}\right)^{n+\frac{n}{n-\alpha}}}dy'\right)^{1-\frac{\alpha}{n}}\le\frac{C}{(r+|x_n-y_n|)^{2-\frac{\alpha}{n}}}.$$

So if we set

$$\Phi(x_n)=\left(\int_{\mathbb{R}^{n-1}}\left|(-\triangle)^{\frac{\alpha}{2}}\varphi(x',x_n)\right|^{\frac{n}{\alpha}}dx'\right)^{\frac{\alpha}{n}}$$

then we have the bound

$$\|\varphi(\cdot,x_n)-\varphi_\varepsilon(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\le C\int_0^\varepsilon\int_{\mathbb{R}}\frac{\Phi(y_n)}{(r+|x_n-y_n|)^{2-\frac{\alpha}{n}}}dy_n dr.$$

Finally, we estimate

$$\int_0^\varepsilon\int_{\mathbb{R}}\frac{\Phi(y_n)}{(r+|x_n-y_n|)^{2-\frac{\alpha}{n}}}dy_n dr.$$

Using dyadic splitting technique, we see

$$\begin{aligned}&\int_0^\varepsilon\int_{\mathbb{R}}\frac{\Phi(y_n)}{(r+|x_n-y_n|)^{2-\frac{\alpha}{n}}}dy_n dr\\&=\int_0^\varepsilon\sum_{k\in\mathbb{Z}}\int_{2^k r<|x_n-y_n|\le 2^{k+1}r}\frac{\Phi(y_n)}{(r+|x_n-y_n|)^{2-\frac{\alpha}{n}}}dy_n dr\\&\lesssim\int_0^\varepsilon\sum_{k\in\mathbb{Z}}\frac{1}{[(2^k+1)r]^{2-\frac{\alpha}{n}}}\int_{B(x_n,2^{k+1}r)}\Phi(y_n)\,dy_n dr\\&\approx\int_0^\varepsilon\sum_{k\in\mathbb{Z}}\frac{2^{k+1}r}{(2^k+1)^{2-\frac{\alpha}{n}}}\frac{1}{r^{2-\frac{\alpha}{n}}}⨍_{B(x_n,2^{k+1}r)}\Phi(y_n)\,dy_n dr\\&\lesssim\left(\int_0^\varepsilon\frac{1}{r^{1-\frac{\alpha}{n}}}dr\right)M\Phi(x_n)\\&\approx\varepsilon^{\frac{\alpha}{n}}M\Phi(x_n).\end{aligned}$$

Hence

$$I(\varepsilon)\le C\|R_1u(\cdot,x_n)\|_{1.\mathbb{R}^{n-1}}\varepsilon^{\frac{\alpha}{nj}}M\Phi(x_n).$$

Now we estimate $II(\varepsilon)$. To estimate this, by fundamental theorem of calculus, we have

$$II(\varepsilon)=-\int_{\mathbb{R}^{n-1}}\int_{x_n}^\infty(R_1u)_{x_n}(x',t)\,\varphi_\varepsilon(x',x_n)\,dtdx'.$$

Then by proposition 2.10, we have

$$\frac{\partial R_j u}{\partial x_i}=\frac{\partial R_i u}{\partial x_j}\quad \text{for all } i,j\in\{1,\dots,n\}$$

and Fubini's theorem to deduce that

$$\begin{aligned}
&-\int_{\mathbb{R}^{n-1}}\int_{x_n}^{\infty}(R_1u)_{x_n}(x',t)\,\varphi_\varepsilon(x',x_n)\,dtdx'\\
&=-\int_{\mathbb{R}^{n-1}}\int_{x_n}^{\infty}(R_nu)_{x_1}(x',t)\,\varphi_\varepsilon(x',x_n)\,dtdx'\\
&=-\int_{x_n}^{\infty}\int_{\mathbb{R}^{n-1}}(R_nu)_{x_1}(x',t)\,\varphi_\varepsilon(x',x_n)\,dx'dt.
\end{aligned}$$

Note that $n\neq 1$. So integration by part gives

$$\begin{aligned}
&-\int_{x_n}^{\infty}\int_{\mathbb{R}^{n-1}}\frac{\partial R_n}{\partial x_1}(x',t)\,\varphi_\varepsilon(x',x_n)\,dx'dt\\
&=\int_{x_n}^{\infty}\int_{\mathbb{R}^{n-1}}R_nu\,(x',t)\,\frac{\partial\varphi_\varepsilon}{\partial x_1}(x',x_n)\,dx'dt.
\end{aligned}$$

Thus by Hölder's inequality, we ahve

$$II(\varepsilon)\le\|R_nu\|_1\sup_{x'\in\mathbb{R}^{n-1}}\left|\frac{\partial\varphi_\varepsilon}{\partial x_1}(x',x_n)\right|.$$

Note by proposition 2.5,

$$\begin{aligned}
\frac{\partial\varphi_\varepsilon}{\partial x_1}(x',x_n)&=\int_{\mathbb{R}^n}\frac{\partial\rho_\varepsilon}{\partial x_1}(y)\,\varphi(x-y)\,dy\\
&=\int_{\mathbb{R}^n}I_\alpha\frac{\partial\rho_\varepsilon}{\partial x_1}(y)\,(-\triangle)^{\frac{\alpha}{2}}\varphi(x-y)\,dy.
\end{aligned}$$

Note

$$I_\alpha\frac{\partial\rho_\varepsilon}{\partial x_1}(y)\approx\int_{\mathbb{R}^n}\frac{1}{\varepsilon^n}\frac{\nabla\rho\left(\frac{x}{\varepsilon}\right)}{|x-y|^{n-\alpha}}dx.$$

Then we divide the integral when $|x|\le 2\varepsilon$ and $|x|>2\varepsilon$ by same reason as before in (3.2). So we have

$$\left|I_\alpha\frac{\partial\rho_\varepsilon}{\partial x_1}(y)\right|\lesssim\frac{1}{(\varepsilon+|y|)^{n-\alpha+1}}.$$

Now by same principle,

$$\begin{aligned}
\left|\frac{\partial\varphi_\varepsilon}{\partial x_1}(x',x_n)\right|&\lesssim\int_{\mathbb{R}^n}\frac{\left|(-\triangle)^{\frac{\alpha}{2}}\varphi(y)\right|}{(\varepsilon+|x-y|)^{n-\alpha+1}}dy\\
&\lesssim\int_{\mathbb{R}}\frac{\Phi(y_n)}{(\varepsilon+|x_n-y_n|)^{2-\frac{\alpha}{n}}}dy_n,
\end{aligned}$$

where

$$\Phi(y_n)=\left(\int_{\mathbb{R}^{n-1}}\left|(-\triangle)^{\frac{\alpha}{2}}\varphi(y',y_n)\right|^{\frac{n}{\alpha}}dy'\right)^{\frac{\alpha}{n}}.$$

So dyadic splitting argument as before gives

$$\left|\frac{\partial\varphi_\varepsilon}{\partial x_1}(x',x_n)\right|\lesssim\frac{1}{\varepsilon^{1-\frac{\alpha}{n}}}M\Phi(y_n).$$

So

$$II(\varepsilon) \leq C \|R_n u\|_1 \frac{M\Phi(y_n)}{\varepsilon^{1-\frac{\alpha}{n}}}.$$

Hence

$$\left| \int_{\mathbb{R}^{n-1}} R_1 u(x', x_n) \varphi(x', x_n) dx' \right| \leq C \left[\|R_n u\|_1 \frac{M\Phi(x_n)}{\varepsilon^{1-\frac{\alpha}{n}}} + \varepsilon^{\frac{\alpha}{n}} M\Phi(x_n) \|R_1 u(\cdot, x_n)\|_1 \right].$$

If $\|R_1 u(\cdot, x_n)\|_1 = 0$, letting $\varepsilon \to \infty$, then we can obtain the desired estimate.

If $\|R_1 u(\cdot, x_n)\|_1 \neq 0$, choose $\varepsilon = \frac{\|R_n u\|_1}{\|R_1 u(\cdot, x_n)\|_1}$ so that

$$\begin{aligned} &\int_{\mathbb{R}^n} (R_1 u) \cdot \varphi dx \\ &\lesssim \|R_n u\|_1^{\frac{\alpha}{n}} \left(\int_{\mathbb{R}} \|R_1 u(\cdot, x_n)\|_1 dx_n \right)^{1-\frac{\alpha}{n}} \left(\int_{\mathbb{R}} (M\Phi(x_n))^{\frac{n}{\alpha}} dx_n \right)^{\frac{\alpha}{n}}. \end{aligned}$$

Then by classical maximal theorem, we obtain

$$\left(\int_{\mathbb{R}} M\Phi(x_n)^{\frac{n}{\alpha}} dx_n \right)^{\frac{\alpha}{n}} \lesssim \left(\int_{\mathbb{R}} \Phi(x_n)^{\frac{n}{\alpha}} dx_n \right)^{\frac{\alpha}{n}} \approx \int_{\mathbb{R}^n} \left| (-\triangle)^{\frac{\alpha}{2}} \varphi \right|^{\frac{n}{\alpha}} dx$$

which completes the proof of the claim and hence the theorem.

References

[1] J. Bourgain and H. Brezis, New estimates for the Laplacian, the div-curl, and related Hodge systems, C. R. Math. Acad. Sci. Paris **338** (2004), 539–543.

[2] L. Grafakos, *Classical Fourier Analysis*, 3rd Edition, Springer, 2014.

[3] S. Helgason, *Radon transform*, Birkhauser.

[4] J. Van Schaftigen, *Estimates for L^1-vector fields*, C. R. Math. Acad. Sci. Paris **339** (2004) 181–186.

[5] A. Schikorra, D. Spector and J. Van Schaftigen, *An L^1-type estimates for Riesz potentials*, arXiv:1411.2318.

[6] E. M. Stein, *Harmonic analysis:Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.

[7] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1971.

[8] W. Rudin, *Functional Analysis*, Mcgraw-Hill